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Synchronization of chaos using proportional feedback

T. C. Newell,* P.M. Alsing,† A. Gavrielides,‡ and V. Kovanis§
Nonlinear Optics Center, Phillips Laboratory,
3350 Aberdeen Avenue SE, Kirtland Air Force Base, New Mexico 87117-5776
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We have demonstrated experimentally a proportional-feedback algorithm for the synchronization of chaotic time signals generated from a pair of independent diode resonator circuits. Synchronization was easily obtained and occurred for relative feedback levels between 3% and 8% of the driving voltage. Once established, the synchronization persisted throughout the whole range of the resonator bifurcation diagram without varying the gain of the feedback.

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I. INTRODUCTION

Upon first consideration, the likelihood of synchronizing chaotic signals appears a dismal prospect at best. Chaos is characterized by a dynamical system's sensitive dependence on initial conditions. Small initial deviations between two chaotic signals lead to exponential divergences of the trajectories in a time on the order of the inverse Lyapunov exponent characterizing the orbit. With the inclusion of either internally or externally generated noise, the prospect of synchronizing two chaotic signals seems to pale even further. Despite this naive pessimism, it has been shown that chaotic synchronization of signals can be realized [1]. Demonstrations have been performed in numerical simulations of evanescently coupled semiconductor laser arrays [2] and experimentally in chaotic circuits [1,3,4]. Recently, methods based on the Ott-Grebogi-Yorke (OGY) scheme for controlling unstable periodic orbits [5] have been adapted [6] and extended [7] to handle the stabilization of a chaotic trajectory of one system about a chaotic trajectory of another system.

In practice, the methods to date for synchronizing two chaotic signals have relied extensively on some knowledge of the system. In the earlier work of Carroll and Pecora [1,3] this meant knowledge of the governing dynamical equations so that the system could be divided into subsystems characterized by Lyapunov exponents of the same sign. The subsystem containing the largest positive Lyapunov exponent can be used to drive the other subsystem (containing only negative Lyapunov exponents) into synchronization with a duplicate subsystem. The work of Mehta and Henderson [6] relied on knowledge of the system's equations in order to construct an artificial dynamical system in which errors between the system's output and the desired aperiodic orbit could be evolved. By using OGY based parameter perturbations when the artificial system was near a fixed point, synchronization of the original system to a target chaotic trajectory could be achieved. Lai and Grebogi [7] used a more direct OGY extension in which the parameter perturbations were applied directly to the original dynamical system. Although their OGY based method intrinsically precludes *a priori* knowledge of the dynamical equations for the system, they still required local knowledge of the Poincaré map in order to calculate the necessary parameter perturbation. While this can in principle be performed numerically for a straightforward application of the OGY method to controlling an unstable periodic orbit, it is more problematic to implement experimentally in the case of synchronizing to an arbitrary chaotic trajectory.

In this paper we present an alternative approach for synchronization of chaotic time signals which is also easy to implement experimentally. In the spirit of the occasional proportional-feedback method developed by Hunt and implemented by him in a diode resonator [8] and with Roy *et al.* [9] in a multimode solid-state laser system, we propose a synchronization scheme based on feeding back a proportional amount of the difference between the two chaotic signals to the voltage $V_0 \sin(2\pi f t)$, which drives one of the circuits. We demonstrate this scheme experimentally in a driven diode resonator of the type well studied in the literature [10]. The outline of this paper is as follows: in Sec. II we discuss the theoretical basis for this method, in Sec. III we present the experimental setup and discuss the experimental results. In Sec. IV we summarize our main results and conclude.

II. SYNCHRONIZATION OF CHAOTIC TRAJECTORIES

Before discussing the use of proportional feedback applied to synchronizing chaotic orbits, let us first recall its normal use in the context of controlling unstable periodic orbits. In the now standard OGY method for controlling

*Also at Department of Physics, University of North Texas, Denton, TX 76203-5368; electronic address: newell@arom.plk.af.mil
†Electronic address: alsing@arom.plk.af.mil
‡Electronic address: tom@photon.plk.af.mil
§Electronic address: kovanis@xaos.plk.af.mil

an unstable periodic orbit of a chaotic dynamical system, the system parameter perturbations necessary to achieve control is given by [5]

$$\delta p_n = \frac{\lambda_u}{\lambda_u - 1} \frac{\boldsymbol{f}_u \cdot [\boldsymbol{x}_n - \boldsymbol{x}_F(p_0)]}{\boldsymbol{f}_u \cdot \partial \boldsymbol{x}_F / \partial p}. \tag{2.1}$$

Here $\boldsymbol{f}_u$ is the contravariant eigenvector, with eigenvalue λ_u, of the unstable direction of the Jacobian $J = \boldsymbol{D}_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}, p)|_{\boldsymbol{x}=\boldsymbol{x}_F, p=p_0}$ of a Poincaré surface of section $\boldsymbol{x}_{n+1} = \boldsymbol{F}(\boldsymbol{x}_n, p_0)$, where p_0 is the nominal value of some control parameter p. The quantity $\partial \boldsymbol{x}_F / \partial p = \boldsymbol{D}_p \boldsymbol{F}(\boldsymbol{x}, p)|_{\boldsymbol{x}=\boldsymbol{x}_F, p=p_0}$ is the change in the fixed point with respect to the control parameter evaluated at the fixed point $\boldsymbol{x}_F$ of the map.

The essence of the occasional proportional-feedback scheme [8,9] for controlling unstable periodic orbits with one local positive Lyapunov exponent is to replace the structural prefactors of Eq. (2.1) with an empirically chosen constant α. This constant then multiplies the difference of the signal V from some reference value V_{ref}. The control algorithm is then simplified to a form which is much easier to implement experimentally,

$$\delta p_n = \alpha \, (V_n - V_{\text{ref}}). \tag{2.2}$$

At least for unstable orbits with one local positive Lyapunov exponent, this scheme is reasonable since it is essentially Eq. (2.1). The reference signal V_{ref} selects out the fixed points of the system and the remaining structural factors which have been lumped into the constant α can be found experimentally by *turning the* α *knob* until control has been achieved. Equation (2.2) is not precisely the OGY formula since the former has been used in practice to control other than unstable period-1 orbits [11]. In addition, the OGY formula implicitly assumes that the parameter perturbation is applied for the entire duration between successive crossings of the Poincaré surface of section. In practice, Eq. (2.2) is typically applied for only a fraction of a characteristic period of the system. This period is essentially the mean value between successive crossings of the Poincaré surface of section. There is precedence for this type of control via proportional feedback called adaptive control [12]. It was implemented as a restabilizing algorithm for systems in which sudden parameter changes created chaotic oscillations and not as a method to control unstable periodic orbits.

A straightforward modification of Eq. (2.2) for synchronizing two chaotic signals would be

$$\delta p_n = \alpha \, (V_n^M - V_n^S). \tag{2.3}$$

Here V_n^S is some chaotic slave signal which we wish to synchronize to some other chaotic master signal V_n^M. Equation (2.3) could be understood as utilizing a feedback signal derived from the difference between the proportional-feedback signals for the master and the slave, i.e., $\delta p_n = \alpha \, (V_n^M - V_{\text{ref}}) - \alpha \, (V_n^S - V_{\text{ref}})$.

Such a scheme was recently proposed by Lai and Grebogi [7] in which the proportionality factor α is in fact iterative dependent. In their paper the authors proposed the synchronization of a slave signal $\boldsymbol{y}_n$ to a master signal $\boldsymbol{x}_n$ via the proportional difference

$$\delta p_n = \left. \frac{[\boldsymbol{D}_{\boldsymbol{y}} \boldsymbol{F}(\boldsymbol{y}, p) \cdot \{\boldsymbol{y}_n - \boldsymbol{x}_n(p_0)\} \cdot \boldsymbol{f}_{u(n+1)}]}{-\boldsymbol{D}_p \boldsymbol{F}(\boldsymbol{y}, p) \cdot \boldsymbol{f}_{u(n+1)}} \right|_{\boldsymbol{y}=\boldsymbol{x}, p=p_0}, \tag{2.4}$$

where in the above, only the derivative terms are evaluated at $\boldsymbol{y} = \boldsymbol{x}, p = p_0$. This formula is derived by expanding the chaotic slave orbit $\boldsymbol{y}_n$ locally about the chaotic master orbit $\boldsymbol{x}_n$ and requiring that the next iteration of $\boldsymbol{y}_n$, after falling into a small neighborhood around $\boldsymbol{x}_n$, lie on the stable direction of $\boldsymbol{x}_{n+1}(p_0)$, i.e., $[\boldsymbol{y}_{n+1} - \boldsymbol{x}_{n+1}(p_0)] \cdot \boldsymbol{f}_{u(n+1)} = 0$. Hence this perturbation formula is proportional to the instantaneous difference between the master and slave signals.

Experimental implementation of this algorithm is nontrivial, especially in real time. For chaotic orbits, one needs to calculate the unstable contravariant eigenvector one step forward in time, i.e., $\boldsymbol{f}_{u(n+1)}$. This requires some knowledge of the mapping so that a small circle of points can be propagated forward in time in order to estimate this unstable direction (see [7,13] for details). In addition, the Jacobian of the map and the derivative of the map with respect to the parameter are iterate dependent since they depend on the current value $\boldsymbol{x}_n$ of the master orbit. This structural knowledge of the system would be needed at each instant along the chaotic trajectory in order to maintain synchronization.

Lai and Grebogi demonstrated numerically that the synchronization was tolerant to a small degree of externally generated noise. This is reminiscent of the robustness of the standard OGY algorithm Eq. (2.1) to noise. In their original paper, OGY defined a region about the unstable fixed point about which control could be achieved. If the magnitude of the noise exceeded the radius of this region, control was lost. A similar, although less well defined, region exists for Eq. (2.4) as well, outside which synchronization will be lost.

An OGY-based scheme requires a system control parameter that one can access and perturb. In our experiment, we did not modulate the amplitude V_0, of the wave-form generator, which is the critical system parameter capable of driving the circuits into chaos. The key ingredient in our synchronization scheme is that the feedback signal is proportional to the difference between the master and slave signals. As synchronization is achieved, such a term will tend towards zero. In our experiment we continously added a proportional amount of the instantaneous master-slave signal difference to the voltage $V_0 \sin(2\pi f t)$, which drove the slave circuit, as long as this difference was smaller than some specified amount. The proportionality factor α was taken to be a constant. In contrast to OGY-based methods, it is not known *a priori* that such small perturbations will drive the circuits into synchronization. We empirically determined the range of driving voltages V_0 over which this scheme could maintain synchronization. The advantage of this method, however, is that is is easily implemented, it does not require us to actively modulate the driving voltage V_0, and synchronization can be achieved with very small relative feedback levels.

In Sec. III we discuss the implementation of this synchronizing proportional feedback algorithm to a diode resonator circuit and find that it works remarkably well. In a forthcoming paper [14] we analyze a dynamical model for our system and show how the synchronization of the two chaotic systems can be achieved by generalizations of these ideas involving the addition of different types of perturbations to the slave system.

III. EXPERIMENT AND RESULTS

A. Setup

The basic idea behind the synchronization experiment is depicted in Fig. 1. From the chaotic attractor of a diode resonator circuit we select out an arbitrary chaotic signal which we designate as the master signal $V^M(t)$. From another identical diode resonator circuit operating under the same conditions we wish to select another chaotic signal, designated as the slave $V^S(t)$, and synchronize it to $V^M(t)$. We accomplish this by measuring the difference between the two signals and feeding back a time varying, proportional amount $\alpha[V^M(t) - V^S(t)]$ to the voltage which drives the slave resonator. The quantity α is to be determined experimentally by adjusting the gain of an amplifier which acts on the difference voltage until the two chaotic signals lock together. In the experimental implementation of the synchronizing proportional-feedback scheme we used the real time signal difference between the master and slave as opposed to their difference measured on some surface of section, as implied by Eq. (2.3).

A block diagram of the experimental setup is shown in Fig. 2. Each diode resonator consists of a 1N4004 diode, a 33-mH inductor, and a 90-Ω resister in series. Both circuits were driven by the same Wave Tek 166 wave-form generator. The master signal V^M and the slave signal V^S were defined as the voltage drop across the respective resistors. Signals were recorded with a Tektronix RTD710A digitizer. Under conditions of no applied feedback, both master and slave signal were approximately 40 mV rms. The master resonator was driven solely by the wave-form generator with a sine wave of frequency 80.5 kHz and amplitude V_0 of 2.92 V rms. This drove the master resonator into the chaotic regime just below the period-3 window which occurred at a driving voltage of 3.0 V rms (see arrow in Fig. 3). In Fig. 4 we have plotted the first-return map for the master signal. This graph was obtained from a long time series recording of the master signal by plotting the $(m + 1)$st peak against the mth peak. The thinness of the attractor indicates that it is nearly one dimensional. However, the extra branch in the attractor and the jump in the size of the attractor at the transition to period three are manifestations of a two dimensional character of the system. Similar observations have appeared in the literature for the diode resonator [15,16].

In addition to being driven at $V_0 \sin(2\pi ft)$, the voltage applied to the slave resonator was modified by the addition of a time varying feedback signal $\delta V_f(t) = \alpha[V^M(t) - V^S(t)]$. This feedback signal was obtained by amplifying the voltage difference between the master and slave oscillators and passing this through a window comparator centered on 0 V. If $\delta V_f(t)$ was inside the range of the adjustable window comparator, an analog switch was closed and $\delta V_f(t)$ was added to the slave driving voltage through a summing amplifier. If $\delta V_f(t)$ fell outside the range of the window comparator, the analog switch remained open and no feedback signal was applied.

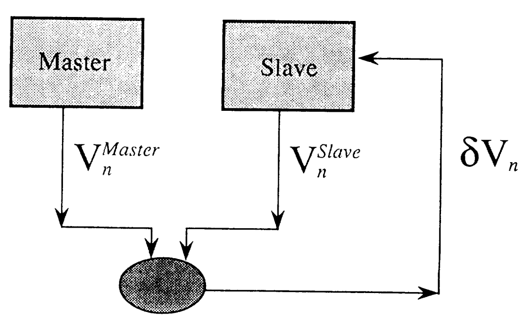

FIG. 1. A schematic illustration of the strategy for synchronizing two almost identical chaotic circuits. The output of the master and the slave circuit is fed into a comparator and the resulting difference is amplified and fed back into the slave circuit in order to induce synchronization.

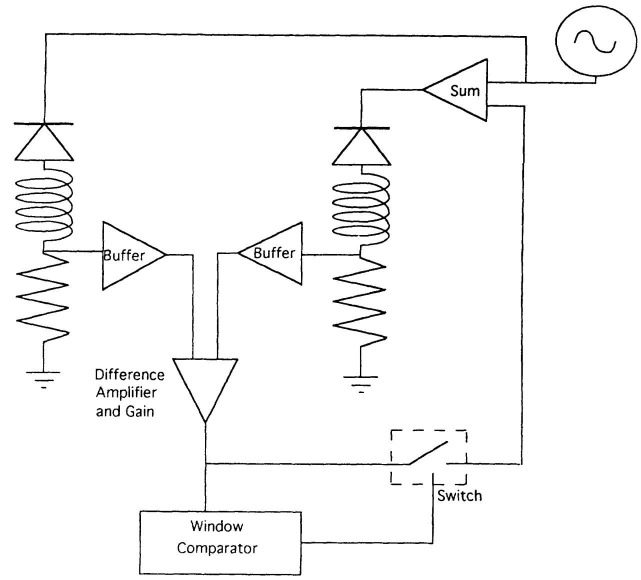

FIG. 2. A block diagram of the experimental apparatus. The setup consists of two identical diode resonators each composed of a 1N4004 diode, 33-mH inductor, and a 90-Ω resistor. Both circuits are driven by the same Wave Tek 166 function generator at 80.5 kHz to ensure that their basic characteristics are the same as much as possible. The chaotic current of each circuit is converted into a voltage and fed into a difference amplifier which is then passed onto a window comparator. If the amplified difference between the two chaotic signals is within an adjustable window of the comparator, an analog switch is closed and the feedback is then added to the slave driving voltage.

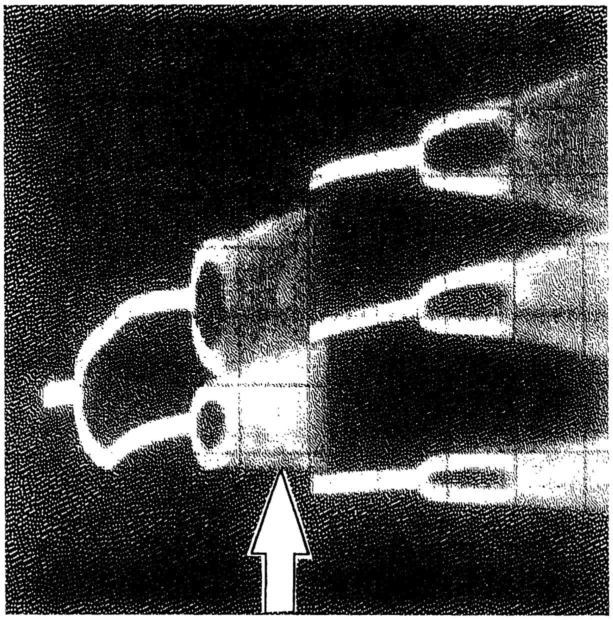

FIG. 3. An oscilloscope trace of the bifurcation diagram for a single diode resonator displaying the period doubling route to chaos. The abscissa is the amplitude of the driving sine wave V_0 while the ordinate is the voltage converted current across the diode (arbitrary units). The arrow indicates the master circuit driving voltage of 2.92 V rms for which the majority of experiments was performed.

B. Observations

Under conditions of zero feedback, the driving voltage for the master and slave circuits were on average identical. Slight variations in the slave driving voltage did exist due the unavoidable noise involved in passing it through the summing amplifier. In Fig. 5(a) we show an oscilloscope trace of $V^M(t)$ versus $V^S(t)$. The resulting wide parallelogram region of phase space demonstrates that each signal was executing a different chaotic trajectory. The 45° slope of the figure is due to the chaotic peaks being approximately in phase as each signal followed the driving voltage. In Fig. 6(a) we plot the raw (unamplified) signal difference $V^M(t) - V^S(t)$.

In order to achieve synchronization, feedback had to be applied for a minimum duration of 1 μs. When the slave signal synchronized to the master, the slave driving voltage dropped to 2.83 V rms. This implied a feedback of approximately 3.5% relative to the master driving voltage of 2.92 V rms. The oscilloscope trace of $V^M(t)$ versus $V^S(t)$ in Fig. 5(b) clearly demonstrates that synchronization has been achieved. Below this feedback level we were unable to force the signals to synchronize. During synchronization the absolute value of the signal difference $|V^M(t) - V^S(t)|$ fluctuated in time, never exceeding 2.5 mV, as shown in Fig. 6(b). We used a gain α of approximately 40 so that the value of the feedback signal $\delta V_f(t)$ never exceeded ± 100 mV. Once the signals were locked together, the comparator window became superfluous since the amplified signal difference was always

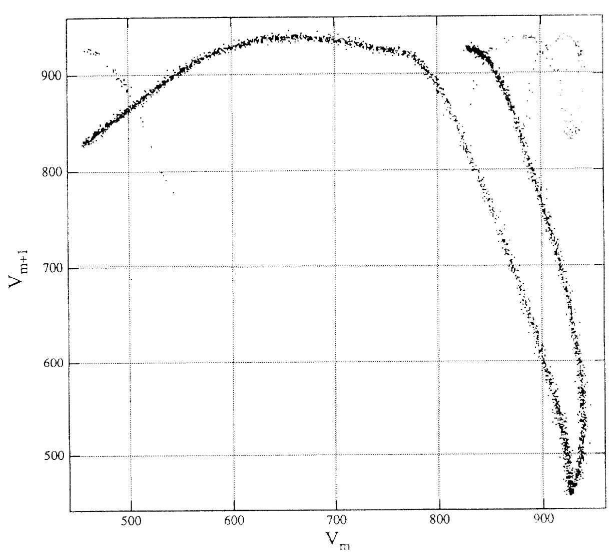

FIG. 4. The first return map for the master circuit driven at 2.92 V rms. This graph was constructed from a long time series recording of the master signal by plotting consecutive pairs of peak values.

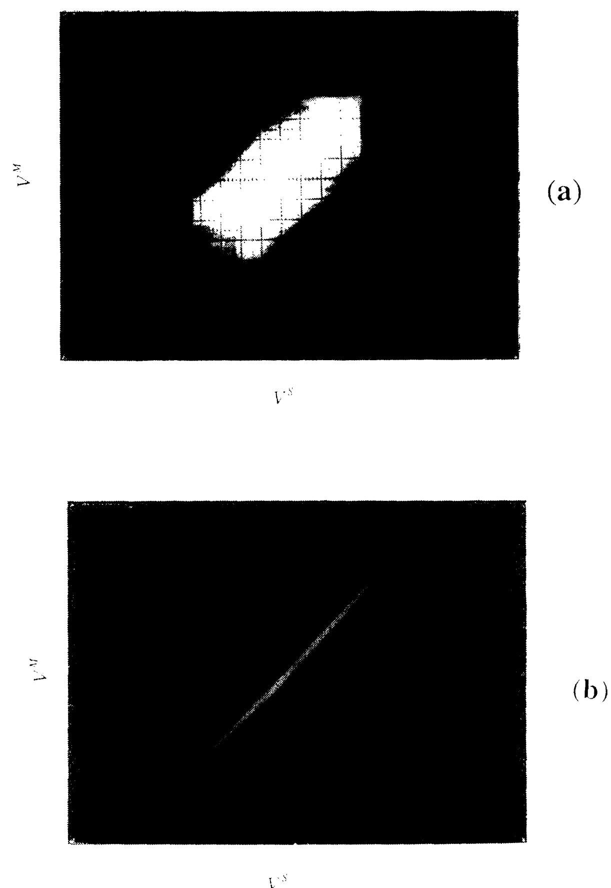

FIG. 5. (a) Oscilloscope trace phase portrait $V^M(t)$ versus $V^S(t)$ of the chaotic circuits when no feedback is applied; (b) the phase portrait when the feedback is applied. The thinness of the trace indicates almost perfect synchronization.

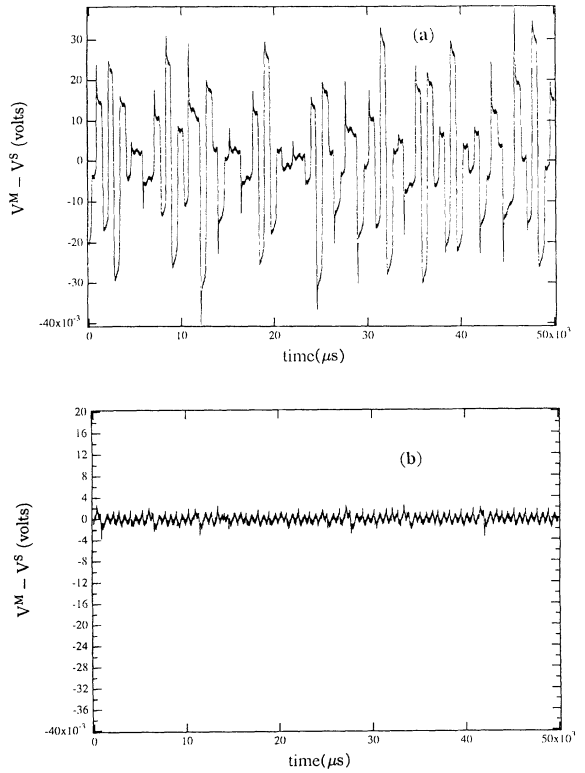

FIG. 6. The master-slave signal difference $V^M(t) - V^S(t)$, (50-MHz sampling rate): (a) no feedback applied, (b) feedback applied. The maximum signal difference is reduced by over a factor of 10 when the signals are synchronized.

within the window comparator, implying that the analog switch then remained continuously closed. The purpose of the comparator window was to ensure that we were not feeding back arbitrarily large perturbations to the slave driving voltage. However, the use of a window comparator was not essential for achieving synchronization. We found that the signals would still synchronize in an experimental arrangement in which the window comparator and the analog switch were removed and the amplified signal difference was fed directly into the slave driving voltage. Again the relative feedback level was approximately 3.5%. We also found that, regardless of the use of a window, the signals could not be locked together if the feedback exceeded 250 mV, corresponding to a relative feedback level of 8.5%.

We observed that once synchronization had been achieved for a given relative feedback level, the driving voltage V_0 could be varied across the full range of the resonator bifurcation diagram (Fig. 3), without varying the gain, and the synchronization would continue to persist. Thus synchronization could be maintained as the signals were changed freely between chaotic and periodic orbits.

As mentioned above, once synchronization was achieved the amplified signal difference always remained within the window comparator. This implied that the analog switch remained closed so that feedback was applied continuously. In another experiment, we chopped the feedback signal in time once synchronization was achieved so that the feedback was turned off periodically. The largest interruption interval in which the feedback signal could be turned off and synchronization was still maintained was 10 μs.

Finally, great care was taken to ensure that the two diode resonators were constructed as identically as possible and operated under similar driving voltages. It is noteworthy to point out that synchronization could still be achieved when the resonator circuits differed slightly from each other due to differences in the various circuit components. In addition synchronization could also be achieved when there existed noticeable differences between the master and slave driving voltages. In these cases the synchronization was somewhat degraded in the sense that the wave form of the slave signal showed noticeable differences from that of the master signal. A comparison of $V^M(t)$ versus $V^S(t)$ similar to Fig. 5(b) revealed a straight line which grew thicker towards the higher voltage end. For slightly dissimilar resonator circuits or nearly identical operating conditions, the chaotic trajectory content of the attractors is different for the two circuits. However, the synchronization employed here allows for successful synchronization of similar chaotic orbits.

IV. CONCLUSIONS

We have demonstrated experimentally that two chaotic signals generated by separate but identical diode resonators could be synchronized by a simple proportional-feedback algorithm. To achieve synchronization, the feedback had to be applied for a minimum duration of 1 μs. The synchronization was maintained by applying relative feedback levels between 3.5% and 8.5%. Once synchronization was established, it could be maintained as the driving voltage was altered while the feedback gain was held constant. We found this synchronizing proportional-feedback scheme more robust and easier to implement than other traditional synchronization methods. The combination of this synchronization scheme with current chaotic controlling algorithms could potentially find use in many applications including communications and chaotic lasers.

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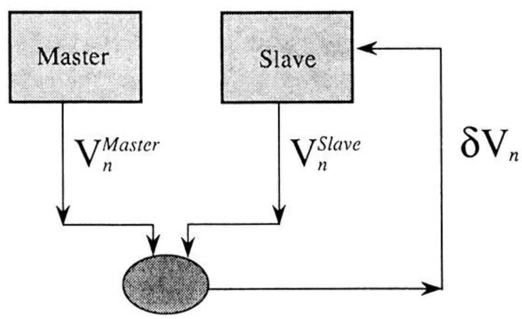

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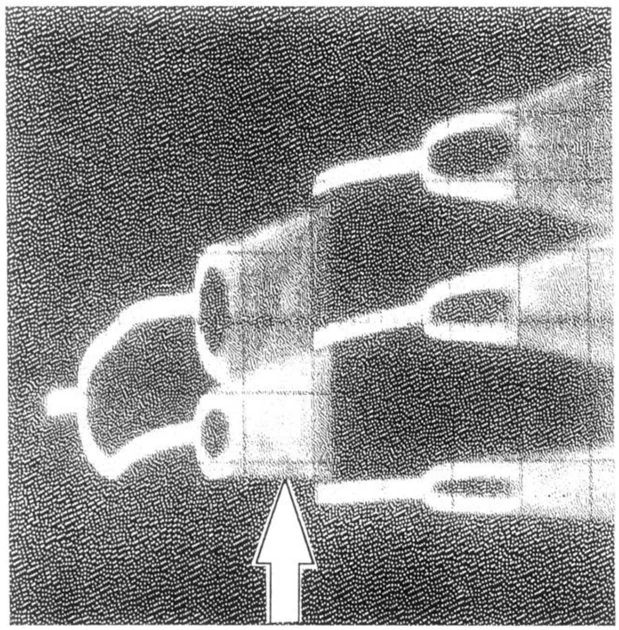

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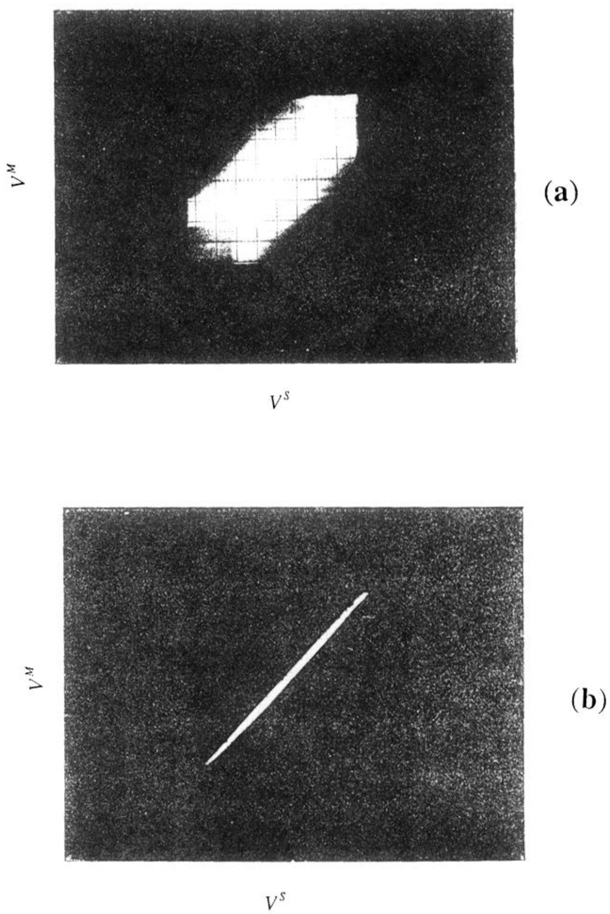

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